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# ALL 2-MANIFOLDS HAVE FINITELY MANY MINIMAL TRIANGULATIONS

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ABSTRACT

A triangulation of a 2-manifold $M$ is said to be minimal provided one cannot produce a triangulation of $M$ with fewer vertices by shrinking an edge. In this paper we prove that all 2-manifolds have finitely many minimal triangulations. It follows that all triangulations of a given 2-manifold can be generated from the minimal triangulations by a process called vertex splitting.

## 1. Introduction

A well-known theorem of Steinitz [4] states that the triangulations of the 2-sphere can be generated from the boundary of the tehrahedron by a process called vertex splitting. In a previous paper [2] the authors showed that for any orientable 2-manifold $M$ there exists a finite set of triangulations such that all triangulations of $M$ can be generated from them by vertex splitting. In this paper we extend this result to all 2-manifolds and we show that our algebraic arguments in [2] can be replaced by a simpler combinatorial argument.

## 2. Definitions and notation

All *manifolds* in this paper are compact 2-dimensional manifolds.

By a *circuit* in a triangulation $T$ of a manifold $M$ we mean a sequence of edges $e_1, e_2, \ldots, e_k$ such that $e_i \cap e_{i+1}$ is a vertex for $i = 1, \ldots, k-1$, $e_k \cap e_1$ is a vertex and other intersections of the $e_i$'s are empty. If a circuit has exactly $k$ edge it will be called a *k-circuit*.

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The intersections of consecutive edges will be called the *vertices* of the circuit. We shall often denote a circuit by listing the vertices in the order that is induced by the order of the edges. We shall also denote an edge with vertices $x$ and $y$ by $xy$.

Let $xy$ be an edge of a triangulation $T$ of a manifold $M$. Let $S$ be the union of all triangles of $T$ meeting $x$ or $y$ and let $B$ be the boundary of $S$. Let $V$ be a set consisting of a point $v$ and all triangles of the form $vst$ where $st$ is an edge of $B$. If we obtain a triangulation $T_1$ of $M$ by replacing $S$ with $V$ we say that $T_1$ is obtained from $T$ by *shrinking* $xy$ and that $T$ is obtained from $T_1$ by *splitting* $v$. The edge $xy$ is called a *shrinkable edge*.

A triangulation $T$ of a manifold $M$ is called *minimal* if and only if no edge is shrinkable. Clearly all triangulations of $M$ can be generated from the set of minimal triangulations by vertex splitting.

A 3-circuit in a triangulation $T$ of $M$ will be called *planar* provided it bounds a subset of $M$ that is a cell, otherwise it will be called *nonplanar*. In [2] the authors showed that if $M$ is not the sphere then a triangulation is minimal if and only if each edge belongs to a nonplanar 3-circuit.

If we have a graph $G$ embedded in a manifold $M$ then the *faces* of $G$ are the connected components of $M - G$. Since $M$ is locally Euclidean, each edge of $G$ is locally 2-sided. If, locally, both sides of an edge $e$ belong to the same face $F$ we shall say that $e$ is a *1-sided edge* of $F$, otherwise $e$ will be called a *2-sided edge*. By the *number of edges* of $F$ we shall mean the number of 2-sided edges in the boundary of $F$ plus twice the number of 1-sided edges in the boundary of $F$.

For any subset $A$ of $M$, we denote by $\bar{A}$ the topological closure of $A$.

## 3. The main theorems

In [2] our proof that there were finitely many minimal triangulations was done by induction on the genus. In that argument, when the genus was reduced we then had triangulations which were not necessarily minimal. We reduced these to minimal triangulations by shrinking edges. The proof of the theorem then depended on showing a bound on the number of these shrinkings, that depended only on the genus.

The bound on the number of shrinkings was a consequence of a bound on the number of simple pairwise nonhomotopic curves meeting only at a base point, that can exist in an orientable manifold of a given genus.

We shall now show how to obtain such a bound for all 2-manifolds.

THEOREM 1. *Let $M$ be a 2-manifold other than the sphere or projective* *plane. Let $E = \{e_1, \ldots, e_n\}$ be a family of simple closed curves in a 2-manifold $M$, satisfying*

$H_1$: *$e_i \cap e_j = \{v\}$, if $i \neq j$,*

$H_2$: *$e_i$ is not homotopic in $M$ to $e_j$, if $i \neq j$,*

$H_3$: *all $e_i$ are homotopically nontrivial in $M$.*

*Let $G$ be the graph represented by $E$. Then $E$ can be extended to a family $E' = \{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+k}\}$, also satisfying $H_1 - H_3$, and representing a graph $G'$ in which each face has at least 3 edges.*

We will begin with the followig lemma, which will provide the basis for an inductive proof of Theorem 1.

LEMMA 1. *Let $F$ be an open surface, having genus $g > 0$. Let the boundary of $F$ be the wedge product of simple closed curves $e_1 \vee e_2 \vee \cdots \vee e_r$, $r \geqq 1$, such that $e_i \cap e_j = \{v\}$, for $i \neq j$. Then there exists a simple closed curve $e_*$ in $\bar{F}$, such that*

(i) *$e_* \cap e_i = \{v\}$, for $i = 1, 2, \ldots, r$.*
(ii) *$e_*$ is not homotopic in $\bar{F}$ to $e_i$, for $i = 1, 2, \ldots, r$.*
(iii) *$e_*$ is homotopically nontrivial in $\bar{F}$.*

PROOF. If $F$ is orientable, then it is homeomorphic to a sphere with $g$ handles, and $r$ boundary components. It follows from the classification theorem for orientable surfaces [see, for example, Massey's *Algebraic Topology*] that the fundamental group of $\bar{F}$ has $2g + r$ generators, which can be represented by the curves $e_1, \ldots, e_r$, plus $2g$ additional closed curves based at $v$. Examination of the single relation between these generators shows that (ii) is satisfied. Therefore there is at least one such curve, $e_*$. If $F$ is nonorientable, then the fundamental group has $g + r$ generators, which can be represented by the curves $e_1, \ldots, e_r$, plus $g$ additional closed curves based at $v$. This proves the lemma.

PROOF OF THEOREM 1. Let $F_i$ be a face of $G$, whose boundary consists of less than three edges, and assume that $F_i$ is a face of largest genus $g \geqq 0$, having this property.

*Case* 1. $F_i$ has exactly one boundary component, $\partial F_i = e_n$, which is counted as one edge of $F_i$ (i.e. $e_n$ is a 2-sided edge of $F_i$).

If $g = 0$, then $F_i$ is an open disc, so $e_n$ is nullhomotopic in $F_i$, and hence also in $M$. This contradicts $H_3$.

If $g > 0$, then $F_i$ is an open surface, possibly nonorientable, having $\partial F_i = e_n$. By Lemma 1 there exists a simple closed curve $e_*$ in $F_i$ such that

(iv) $e_* \cap e_n = \{v\}$,

(v) $e_*$ is not homotopic in $\bar{F}_i$ to $e_n$,

(vi) $e_*$ is not nullhomotopic in $\bar{F}_i$.

We will show that the family $\{e_1, \ldots, e_n, e_*\}$ satisfies $H_1$–$H_3$.

If $e_*$ is nullhomotopic in $M$ there is an open disc $D$ in $M$, such that $\partial D = e_*$. By (vi) $D$ is not contained in $\bar{F}_i$, so that $D \cap (M - \bar{F}_i) \neq \varnothing$. Since $D$ meets each of the open sets $F_i$ and $M - (\bar{F}_i)$, it must meet their intersection, so $D \cap e_n \neq \varnothing$. Furthermore, $D \cap e_n$ is an open subset of $e_n$, in the relative topology. If $D \cap e_n \neq e_n - \{v\}$ then $(\partial D) \cap e_n = e_* \cap e_n \neq \{v\}$, contradicting (iv). If $D \cap e_n = e_n - \{v\}$ then $e_n \subset \bar{D}$, and therefore $e_n$ is null homotopic in $M$, contradicting $H_3$.

If $e_*$ is homotopic in $M$ to some $e_j$, for $j = 1, \ldots, n-1$ then there is an open surface $S$ in $M$ such that $\partial S = e_j \vee e_*$. Since $S$ intersects both $F_i$ and $M - \bar{F}_i$, it must intersect $\partial F_i = e_n$. $\bar{S}$ is in fact homeomorphic to the reduced suspension of $e_*$, from which it follows that if $e_n$ is contained in $\bar{S}$ then $e_n$ is homotopic in $M$ to $e_j$. If $e_n$ is not contained in $\bar{S}$, the same argument as in the previous case implies $e_j \cap e_n \neq \{v\}$.

Finally we will show that $e_*$ cannot be homotopic in $M$ to $e_n$. Assuming such a homotopy exists, there is an open surface $S$ in $M$, with $\partial S = e_n \vee e_*$.

If $S \cap F_i$ and $S \cap (M - \bar{F}_i)$ are both nonempty, then so is $S \cap e_n$. But $S$ is open, and $e_n \subset \partial S$, which is a contradiction. If $S \cap (M - \bar{F}_i) = \varnothing$, then $S \subset \bar{F}_i$, contradicting (v).

*Case* 2. $F_i$ has exactly one boundary component $\partial F_i = e_n$, which is counted as two edges of $F_i$ (i.e. $e_n$ is a 1-sided edge of $F_i$).

If $g = 0$ then $\bar{F}_i$ is a closed disc with its boundary identified so that $\partial F_i$ is a 1-sided curve, thus $\bar{F}_i$ is a projective plane and $\bar{F}_i = M$, thus $M$ is a projective plane.

If $g > 0$ then the argument in Case 1 applies.

*Case* 3. $F_i$ has exactly two boundary components, $e_{n-1}$ and $e_n$.

If $g = 0$, then $F_i$ is an open disc having boundary $\partial F_i = e_{n-1} \vee e_n$. In this case $e_{n-1}$ is clearly homotopic to $e_n$ in $\bar{F}_i$, and hence also in $M$.

If $g > 0$, then $F_i$ is a open surface (*possibly nonorientable*) with $\partial F_i = e_{n-1} \vee e_n$. Again by Lemma 1 there exists a curve $e_*$ in $\bar{F}_i$ satisfying (iv)–(vi).

Exactly as in Case 1, we can show that the family of curves $\{e_i, \ldots, e_n, e_*\}$ also satisfies $H_1$–$H_3$.

The preceding arguments show that the family $\{e_1, \ldots, e_n, e_*\}$ satisfy the hypotheses $H_1$–$H_3$. The process of adding the curve $e_*$ to the family $E$ replaces the face $F_i$ by one or two faces of genus $g' < g$. Since $M$ has a finite number of faces the procedure terminates after a finite number of steps, which proves that all faces of maximum genus having one or two boundary components can be eliminated.

COROLLARY 1. *If $M$ is not the sphere or projective plane, then any family $E$ of homotopically nontrivial simple pairwise nonhomotopic curves meeting at a base point $v$ in $M$ has at most $6g - 3$ members when $M$ is orientable, and $3g$ members when $M$ is nonorientable, where $g$ is the genus of $M$.*

PROOF. We extend $E$ to a family $E'$ satisfying $H_1$–$H_3$ and such that each face has at least 3 edges. Let $G$ be the graph represented by $E'$. Let $G$ have $v$ vertices, $e$ edges and $F$ faces. Standard counting arguments now show that $2e \geqq 3f$, which together with Euler's inequality ($v - e + f \geqq 2 - 2g$) for orientable manifolds and $v - e + f \geqq 1 - g$ for nonorientable manifolds implies that $e \leqq 6g - 3$ for orientable manifolds and $e \leqq 3g$ for nonorientable manifolds.

This corollary fills an omission in [2] where we assumed any such family of curves belongs to a maximal family. Although the existence of the maximal family does follow from the algebraic argument in [2] it was not so stated.

THEOREM 2. *If $M$ is a manifold of genus $g$, then there are finitely many minimal triangulations of $M$.*

PROOF. The proof is like the proof in [2] with only slight differences. We shall point out the differences and refer the reader to [2] for a complete exposition of the argument.

The Theorem is known for $g = 0$ and orientable genus 1 (see [1], [3] and [4]). If $M$ is any other manifold, let $T$ be a minimal triangulation of $M$. We cut $M$ along a nonplanar 3-circuit $xyz$. If the result of this cut is a set (consisting of one or two manifolds with boundary) with two bounding 3-circuits, then the argument is identical to [2]. In the nonorientable case, however, we may obtain a manifold with a bounding 6-circuit $C$ (in the case where $xyz$ has a neighborhood that is a Möbius strip). To this manifold with boundary we attach a cell consisting of all triangles of the form $wab$ where $w$ is a point not in $M$ and $ab$ is an edge of $C$. This produces a triangulation $T_1$ of a new manifold $M_1$ of genus $g$ less than the genus of $M$.

We now produce a minimal triangulation $M^*$ from $M_1$ by shrinking edges. Recall that the only possible shrinkable edges in $M_1$ will be those that do not belong to any nonplanar 3-circuit in $M_1$, and that all edges of $M$ belong to nonplanar 3-circuits.

The only possible shrinkable eges of $M_1$ are thus edges of 3-circuits homotopic to $C$ in $M_1$, edges belonging to nonplanar 3-circuits meeting $x$, $y$ or $z$ in $M$, and edges meeting $w$. In [2] we show that there is a bound depending only on $g$ on the number of edges of the first two types. Since there are just six edges meeting $w$, there is a bound on the number of shrinkable edges.

Since shrinking an edge does not change the homotopy type of any 3-circuit, this puts a bound on the number of edge shrinkings necessary to produce $M^*$.

By induction there are only finitely many combinatorial types for $M^*$. As is shown in [2], it follows that there are only finitely many combinatorial types for $M_1$ and thus also for $M$.

## References

1. D. W. Barnette, *Generating the triangulations of the projective plane*, J. Comb. Theory **33** (1982), 222–230.

2. D. W. Barnette and Allan Edelson, *All orientable 2-manifolds have finitely many minimal triangulations*, Isr. J. Math. **62** (1988), 90–98.

3. S. A. Lavrenchenko, *The irreducible triangulations of the torus*, Geom. Sb. **30** (1987), 52–62 (Russian).

4. E. Steinitz and H. Rademacher, *Vorlesungen über die Theorie der Polyeder*, Springer-Verlag, Berlin, 1934.